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Fluxons and exact BPS solitons in non-commutative gauge theory

Koji Hashimoto

Institute for Theoretical Physics, University of California
Santa Barbara, CA 93106, USA
E-mail: `koji@itp.ucsb.edu`

Abstract: We show that the fluxon solution of the non-commutative gauge theory and its variations are obtained by the soliton generation method recently given by J. A. Harvey, P. Kraus and F. Larsen (`hep-th/0010060`). Although this method generally produces non-BPS solutions of equations of motion, the solutions we obtained are BPS. We give the brane interpretation of these BPS solutions and study their counterparts in the ordinary description by the Seiberg-Witten map.

Keywords: Solitons Monopoles and Instantons, D-branes, Brane Dynamics in Gauge Theories, Non-Commutative Geometry.

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Contents

1. Introduction

Introducing the notion of non-commutative space in string theories have made fruitful and remarkable results in these years. Among them, solitons in various non-commutative theories have played a central role in understanding the physics of non-commutative theories and certain situations of string theories. One of the applications is the string field theories in which introducing the non-commutativity makes it possible to construct D-branes as solitons [1, 2, 3]. On the other hand, the non-commutative field theories are interesting subjects by themselves, especially when they are realized as the low energy description of D-branes in string theories. Solitons in these theories have interpretation of the brane configurations, and using the brane configuration techniques new phenomena such as the non-locality of the non-commutative monopoles [4, 5, 6] and the resolution of the small instanton singularity in the moduli space of the non-commutative instantons [7, 8] have been investigated.

In the sequence of the above study, some exact solutions for finite non-commutativity parameters have been constructed [7], [9]–[18]. Among them, the transformation proposed recently by Harvey, Kraus and Larsen (HKL) [16] is particularly interesting, because it transforms a trivial solution (such as a vacuum solution) into a non-trivial classical solution of equations of motion. We summarize their method briefly here. Let us consider a general action written in terms of various fields ϕ_i which are represented by operators on the the non-commutative space. The equations of motion are

$$\frac{\delta S}{\delta \phi_i} = 0\,. \tag{1.1}$$

The HKL transformation is defined using the "almost" unitary operator U as

$$\phi_i \longmapsto U \phi_i U^\dagger\,, \tag{1.2}$$

where $U^\dagger U = I$, however $UU^\dagger \neq I$. Under this transformation, the equations of motion remain intact:

$$\frac{\delta S}{\delta \phi_i} \longmapsto U \frac{\delta S}{\delta \phi_i} U^\dagger = 0\,. \tag{1.3}$$

Since to show the invariance of the equations of motion one uses only $U^\dagger U = I$, the opposite combination $UU^\dagger$ is not necessarily a unity. Because $(UU^\dagger)(UU^\dagger) = U(U^\dagger U)U^\dagger = UU^\dagger$, the combination $UU^\dagger$ must be a projection operator. Therefore, if we adopt some non-trivial projector $UU^\dagger$, the transformation (1.2) generates new solitons of equations of motion. Note that we have assumed that there is no source term for the field ϕ_i in the action. If the source term is present, then a part of the equations of motion coming from the source term is not invariant under the HKL transformation.

In general, the HKL transformation generates non-BPS solutions. This is because the BPS equations in non-commutative gauge theories generally contain a constant term which becomes a source term in the BPS equations mentioned above. However, in this paper, using a simple trick, we show that the BPS fluxon solution [12, 17] which represents a D-string piercing a D3-brane can be reconstructed by the HKL method. A virtue of this reconstruction is that we can find variations of the fluxon solutions. These solutions have interesting brane interpretation. Using the Seiberg-Witten map [8] which relates the non-commutative description to the ordinary description, we study the width of the fluxon and clarify the reason why there is no corresponding solution in the ordinary description.

2. BPS equation and solution

Let us consider the $(1+3)$-dimensional non-commutative gauge theory with a scalar field Φ. We introduce non-commutativity only in the x^1-x^2 plane: $[x^1, x^2] = i\theta$. The equations of motion in the operator representation are written as

$$[D_\nu, [D_\nu, D_\mu]] + [\Phi, [\Phi, D_\mu]] = 0\,, \tag{2.1}$$

$$[D_\mu, [D_\mu, \Phi]] = 0\,. \tag{2.2}$$

We have defined covariant derivatives $D_\mu \equiv \partial_\mu + A_\mu$ ($\mu = 0, \ldots, 3$) where the gauge field A_μ is anti-hermitean. One recognizes that this equations of motion are that of IKKT IIB Matrix model [19]. When written in terms of operator language, the equations of motion can be expressed always in the form of matrix models. This indicates a close relation between non-commutative gauge theories and matrix models.

Assuming static configurations $\partial_0 = 0$ and no electric field excitation $A_0 = 0$, the above equations of motion are consistent with the following first order BPS equation

$$B_i + [D_i, \Phi] = 0\,, \tag{2.3}$$

where B_i is the magnetic field and $i = 1, 2, 3$. Taking the gauge $A_3 = 0$, the BPS equations become

$$\partial_3 \Phi = -i[D_1, D_2] + \frac{1}{\theta}, \qquad \partial_3 D_1 = -i[D_2, \Phi], \qquad \partial_3 D_2 = -i[\Phi, D_1]. \tag{2.4}$$

Defining $D \equiv (D_1 + iD_2)/\sqrt{2}$ and $\bar{D} \equiv -D^\dagger$, we write these equations in a simple form for the latter convenience as

$$\partial_3 \Phi = [D, \bar{D}] + \frac{1}{\theta}, \qquad \partial_3 D = [D, \Phi]. \tag{2.5}$$

The HKL transformation applied to this theory is

$$D \longmapsto UDU^\dagger, \qquad \Phi \longmapsto U\Phi U^\dagger. \tag{2.6}$$

So as to keep the $A_3 = 0$ gauge, the transformation operator U has to be independent of x^3.

One notices immediately that, though this transformation keeps the equations of motion (2.2) invariant, it changes the BPS equations (2.5). This is simply because the constant term $1/\theta$ exists in eqs. (2.5). This constant term behaves as if it is a source term in the equation. Generally, BPS equations are the first order equations and contain field strengths which are not in commutators. So the BPS equations in the non-commutative space have the constant term which is not invariant under the HKL transformation. This shows that the HKL transformation generates non-BPS solutions of equations of motion in general.

However, in our case, there is a certain method to obtain BPS solutions using the HKL transformation. Note that above BPS equations (2.4) are precisely the Nahm's equations for the non-commutative monopoles [6, 10]. The trick used in the papers was to redefine one of the ingredients as

$$\Phi^{(\mathrm{P})} \equiv \Phi - \frac{z}{\theta}. \tag{2.7}$$

In terms of this $\Phi^{(\mathrm{P})}$, the BPS equations (2.5) do not include the constant term $1/\theta$ and they look as if they are in the commutative space.

$$\partial_3 \Phi^{(\mathrm{P})} = [D, \bar{D}], \qquad \partial_3 D = [D, \Phi]. \tag{2.8}$$

Hence we can apply the HKL transformation on these equations without the constant term.

Before applying the transformation, let us see simple solutions to be transformed. One of them is

$$\Phi^{(\mathrm{P})} = \Phi_0 - \Phi_1 x^3, \qquad D = \sqrt{\Phi_1}\, a, \tag{2.9}$$

where Φ_0 and $\Phi_1 (\geq 0)$ are real constant parameters and a is a creation operator: $a \equiv (x^1 + ix^2)/\sqrt{2\theta}$. Some particular choices of these parameters exhibit interesting

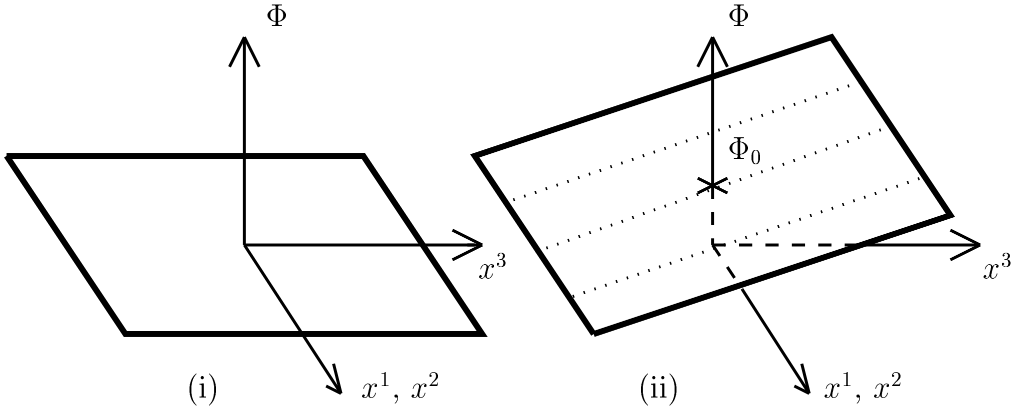

Figure 1: Simple solutions to be transformed: (i) the trivial vacuum, (ii) the smeared D-string with no D3-brane surface.

solutions. First, the choice (i) $\Phi_1 = 1/\theta, \Phi_0 = 0$ provides us with a trivial vacuum solution with $\Phi = 0$, $B = 0$. The second choice (ii) $\Phi_1 = 0$, $\Phi_0 \neq 0$ is interesting. In this case $\Phi = \Phi_0 + z/\theta$ (see figure 1*ii*) and we have a constant field strength $B_3 = -([D, \bar{D}] + 1/\theta) = -1/\theta$. Note that the surface expressed by the scalar Φ has the constant slope which is exactly the same slope as the one that a fluxon solution [12, 17] has. The fluxon solution represents a D-string piercing the D3-brane, therefore our solution (ii) represents a brane configuration that all the world volume is filled with many parallel piercing D-strings. We cannot see the D3-brane! How this situation is possible in the non-commutative theory and impossible in the equivalent ordinary theory will be addressed later.

Then let us perform the HKL transformation on this solution (2.9). Taking the simplest non-trivial transformation $U = \sum_{n\geq 0} |n{+}1\rangle\langle n|$, the result for the choice (i) is

$$\Phi = U\Phi^{(\mathrm{P})}_{\mathrm{original}}U^\dagger + \frac{z}{\theta} = \frac{z}{\theta}P_0\,, \qquad B_3 = \frac{1}{\theta}P_0\,, \tag{2.10}$$

where P_0 is the projection operator onto the state $|0\rangle$. This is precisely the BPS fluxon [12, 17] (see figure 2*i*). Thus we have reproduced the BPS fluxon solution using the HKL transformation. For the choice (ii), the result is

$$\Phi = \Phi_0(1 - P_0) + \frac{z}{\theta}\,, \qquad B_3 = -\frac{1}{\theta}\,. \tag{2.11}$$

Note that the magnetic field is not changed from the one before the HKL transformation. From the configuration of Φ depicted in figure 2*ii*, we interpret this solution as the smeared many parallel D-strings with a single D-string protruded out of them in parallel. This solution is similar to the non-BPS solutions found in ref. [15]: a certain moduli of the solution is corresponding to the transverse separation of the object from the main brane. In our case, the parameter Φ_0 measures the separation

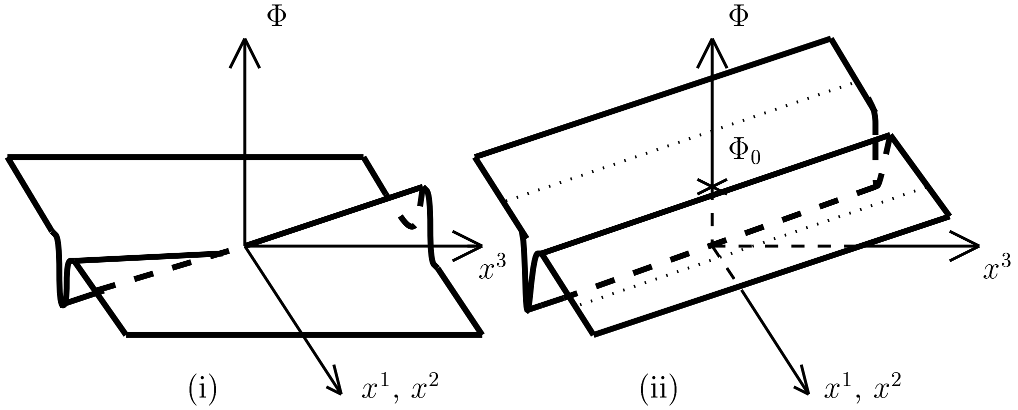

Figure 2: The HKL-transformed configurations: (i) the BPS fluxon, (ii) a single D-string and the smeared parallel D-strings with no D3 surface.

of a single D-string from the other smeared surface of the D-strings. Again, we cannot see the D3-brane surface. To our best knowledge, no similar solution has been found in the ordinary theories.

We can generalize the above construction easily to the non-abelian case. Adopting a simple solution (we are working in U(2) gauge group for simplicity)

$$\Phi^{(\mathrm{P})} = \Phi_1 \; z \; \mathbb{1} + \Phi_0 \; \sigma_3 \,, \qquad D = \sqrt{\Phi_1} \; a \; \mathbb{1} \,, \tag{2.12}$$

after the HKL transformation we obtain a generalized fluxon which represents a D-string piercing two parallel D3-branes.

3. Width of the fluxon and Seiberg-Witten map

The fluxon solution was originally constructed by observing an asymptotic behavior of the non-commutative U(1) monopole solution in ref. [12]. At $x^3 = +\infty$, the non-commutative monopole solution becomes extremely simple and have the form of eq. (2.10). The asymptotic value of Φ is given by the projection operator P_0 which is a Gaussian of the width $\sqrt{\theta}$ whose center is located at the origin of the non-commutative plane (x^1-x^2 plane). Therefore the fluxon has the width of $\sqrt{\theta}$.

According to ref. [8], this non-commutative theory has an equivalent ordinary description with the NS-NS 2-form b-field, not with the non-commutativity. This is the ordinary Dirac-Born-Infeld theory with the b-field, and in this theory a solution corresponding to the non-commutative monopole through the Seiberg-Witten map was constructed [20, 21, 22]. The construction of this solution is due to the performance of the rotation in the target space. In fact, the solution satisfies a non-linear BPS equation, and this rotation in the target space relates the non-linear BPS equation to the linear one,

$$B_i + b_i + \partial_i \Phi = 0 \,, \tag{3.1}$$

where $b_i = \epsilon_{ijk} b_{jk}/2 = b\delta_{i3}$ is the b-field. The solution of this linear equation is

$$\Phi = \frac{1}{r} - bx^3 \,. \tag{3.2}$$

The first term ("spike" of the BIon) shows the D-string ending on the D3-brane [23, 24]. The second term represents a slope of the D3-brane surface. Performing on the above linear solution (3.2) the rotation in the target space so that the D3-brane surface become horizontal, one obtains the solution of the non-linear BPS equations [20, 21, 22]. This configuration with the horizontal D3-brane corresponds to the non-commutative monopole of ref. [12] through the Seiberg-Witten map.

As is seen in the following, in the ordinary description, we cannot obtain a solution representing a D-string piercing the D3-brane. If one wants to pierce the D3-brane, one has to add $-1/r$ to the above solution (3.2). This $-1/r$ term represents the D-string elongating in the $\Phi \to -\infty$ direction. However, this term cancels the first term in (3.2) and the whole spike vanishes.

Another argument is as follows: The fluxon solution is obtained by seeing the asymptotic behavior ($x^3 \to \infty$) of the non-commutative monopole. If one see the corresponding asymptotic behavior of the solution of the ordinary side in refs. [21, 22], it is easy to find that at the positive infinity of x^3 the solution becomes singular. The width of the BIon is getting thinner and thinner in this asymptotic region.

These arguments show that the fluxon solution in the ordinary description does not exist. However, if one believes the validity of the Seiberg-Witten map, the fluxon can be mapped to some configuration in the ordinary description. Then what happens to the Seiberg-Witten map?

The hint for answering this question is in the solution of the choice (ii) above (figure 1*ii*). That solution consists simply of many parallel D-strings, no D3-brane. Let us pay attention to the field strength of that solution. As in ref. [8], the Seiberg-Witten map can be exactly solved for constant field strength. Particularly, when $B_3 = -1/\theta$, it was shown that there is no corresponding ordinary description: $F_{\text{ordinary}} = \infty$. This is the case for the choice (ii). Thus there is no solution like this (ii) in the ordinary description. In this sense, this solution (ii) is proper to the non-commutative gauge theory.

Now, let us see the Seiberg-Witten map of the monopole solution (3.2). As in the same manner, when $B_3 + b = 0$ in the ordinary description, the Seiberg-Witten map becomes singular and the non-commutative field strength diverges [8]. At the infinity $r = \infty$ the solution (3.2) satisfies $B_3 + b > 0$ (we have assumed $b > 0$ for simplicity). So we cannot reach the region $B_3 + b < 0$ in the non-commutative description because at the point $B_3 + b = 0$ the Seiberg-Witten map becomes singular.

The region $B + b < 0$ is almost a ball with a radius $1/\sqrt{b}$ in the world volume. At the surface of this ball the non-commutative field strength is diverging. So this radius corresponds to the size of the flux tube in the non-commutative description.

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Using the essential relation $b \sim \theta^{-1}$ [8] in the Seiberg-Witten map, we expect that the width of the fluxon is $\sqrt{\theta}$. This is in agreement with the explicit solution of the fluxon (2.10).

This argument shows also that the non-commutative monopole solution of ref. [12] corresponds to only a part of the D3-brane surface of the solution in the ordinary description:[1] in the region $B + b < 0$ of the solution (3.2) there is no non-commutative counterpart which is properly defined also at $r = \infty$.

4. General argument and discussion

In section 2, we have applied the HKL transformation to the BPS equations of the non-commutative monopoles. It needs the trick of redefinition of the scalar field to apply the transformation.

Usually BPS equations are the first order equations, and therefore contain terms consisting merely of field strengths F. On non-commutative space the field strength is defined using the commutator of the derivatives which now gives a constant term. Due to this constant term in the field strength, the BPS equations are not left intact under the HKL transformation, though the equations of motion are invariant on the contrary because they are usually the second order equations. Taking a commutation of F, then the troublesome constant terms is always dropped. Hence, usually the HKL transformation does not generate solutions satisfying BPS equations.

A natural question is the following: when can we apply the HKL transformation to BPS equations? In section 2, we have used the trick to absorb the constant term by noting that the Nahm's equation is in the same form as the BPS equations of the non-commutative monopoles (2.4). This coincidence is because these two are related with each other by a non-commutative analogue of the Nahm's transform [25]. (The Nahm's equation can be understood as a BPS equation on D-string worldsheet gauge theory [26, 27]. To deal with the non-commutative monopoles in string theory, one needs to put D-strings ending on parallel D3-branes, in the b-field [4]. The above Nahm's transform is possibly understood as T-dualities with the b-field in terms of string theory (see, for example, ref. [28]), which exchange the roles of D-strings and D3-branes.) The Nahm's equation parameterize the moduli space of monopoles. The trick in section 2 was used to show that the non-commutativity does not affect

[1] In the above argument, we have applied naively the result of the Seiberg-Witten map of the constant field strength to the non-constant case. However, we believe that the qualitative argument survives even in the case of the non-constant field strength. Another assumption used above is that we can rotate the D3-brane surface in the target space even in the non-commutative theories, as in the case of ordinary Dirac-Born-Infeld theory [22]. This assumption is not negligible: though the global rotation in the target space has to be accompanied by the diffeomorphism, in the non-commutative theories how the diffeomorphism acts on gauge fields is not well defined. There exists the ordering ambiguities. This difficulty is precisely the same as the one of non-abelian Dirac-Born-Infeld theory.

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the moduli space [6, 21]. Hence, this suggests that if the non-commutativity does not change the moduli space then we can use the HKL transformation to obtain BPS solutions.

This argument is consistent with the self-dual instanton solution [15] which is BPS but can be obtained by the HKL transformation. Note that in this case the non-commutativity $\theta^{\mu\nu}$ is also self-dual, thus the self-dual equation describing this non-commutative instanton is not modified. Consistently with the Nahm's transform, the ADHM equation is not modified by the self-dual non-commutativity parameter [8], thus the moduli space of this instanton is not changed and includes a small instanton singularity [18].

This shows that the BPS solution generated by the HKL transformation has the moduli space which is precisely the same as the ordinary version of that.

The above argument is applied only for the BPS equations which can be obtained by the dimensional reduction of the self-dual instanton equation in 4-dimensional Yang-Mills theory. One of the other examples of such theories is the BPS non-abelian vortex described by the Hitchin equation. The non-commutative version of this BPS equation contains a constant term explicitly which cannot be removed by any field redefinition. So this BPS vortex cannot be treated by the HKL transformation.

One of the other types of the BPS equations is on the vortex of the $(1+2)$-dimensional non-commutative Abelian-Higgs model [14, 16, 29]:

$$[D, \bar{\phi}] = [\bar{D}, \phi] = 0\,, \qquad B = \phi\bar{\phi} - \phi_0^2\,. \tag{4.1}$$

Here $|\phi| = \phi_0$ is the bottom of the potential for the complex scalar field ϕ, and only for the special choice of the coefficient of this potential we can achieve the saturation of the BPS bound with the above equations [29]. Since the definition of the magnetic field in the non-commutative space is $B = [\bar{D}, D] - 1/\theta$, we observe that if and only if $\theta = 1/\phi_0^2$, the constant term in eq. (4.1) vanishes and we can apply the HKL transformation to obtain BPS solutions. We shall not write the explicit solutions here because a similar solution has already been obtained in ref. [14, 16].

Not only the fluxons, but also the BPS non-commutative monopole solution itself [10, 17] is possibly generated by the HKL transformation using the trick of section 2. The non-commutative monopole solution depends on x^3, thus accordingly the soliton-generating operator U should be x^3-dependent. In section 2 we have chosen a x^3-independent U so as to keep the $A_3 = 0$ gauge. However, we can choose another prescription of the application of the HKL transformation: assume that A_3 is not transformed by any U. Then the remaining equations (2.8) is invariant under the HKL transformation if

$$[\Phi, U^\dagger \partial_3 U] = [D, U^\dagger \partial_3 U] = 0\,. \tag{4.2}$$

In this way one can generate x^3-dependent BPS solutions. However, the above constraint (4.2) on $U(x^3)$ is turned out to be difficult to be solved even for a specific choice of Φ and D (such as the vacuum solution). We leave this issue for the future study.

Note added. After the completion of this paper, we became aware of the paper [30] which contains the overlapping results.

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